**An atoms-to-mesoscale approach to ice-vapor surface dynamics with a quasi-liquid interface**

Steven Neshyba1, Tia Böttger1, Rohan Crossland1, Spencer Racca-Gwozdzik1, Ella Slattery1, Maximilian Bloom, Noah Zimmer1, Penny M. Rowe2, and Jacob Price1

1University of Puget Sound, Tacoma Washington USA

2North West Research Association, Redmond Washington, USA

**Abstract**

*We explore the hypothesis that a key factor in determining the dynamics and morphology of faceted ice-vapor surfaces is the quasi-liquid layer that forms at this interface at temperatures above . We do so by modeling the ice surface as a system of reaction-diffusion equations in which the time scales of quasi-liquid freezing and melting, horizontal diffusion, and exchanges with the vapor phase are made explicit. Model parameterizations are informed by atomistic (molecular dynamics) simulations, mesoscale simulations of the vapor field around growing and ablating ice crystals, and quantitative ice surface morphologies derived from scanning electron microscope experiments. The outcome is a more unified, predictive, and experimentally grounded picture of the dynamics and morphology of faceted ice-vapor surfaces than has previously been presented.*

1. **Introduction**

Numerous frameworks for thinking about ice surface morphology and dynamics during vapor depositional growth have been presented (Harrington and Pokrifka [2021] provide an excellent review of these). These frameworks range from empirical to atomistic, and are often focused on specific conditions and crystal surface types. The so-called capacitance model, for example, makes the assumption that the efficiency with which an ice surface captures water vapor that strike it is 100%, rendering it applicable to rough ice surfaces, but not faceted ones.

Of these, models rooted in atomistic structure and processes are appealing because of the evident connection between molecular structure on the one hand, and high-resolution microscopic observations (such as scanning electron microscopy) on the other; a familiar example is that crystals of water ice are understood to owe their hexagonal shape to the hexagonal structure of the unit cell. Another is the Burton-Cabrera-Frank model of crystal growth (also called classical nucleation theory), which takes into account gas-phase diffusion and surface kinetic effects: when a gas-phase molecule (e.g., a water molecule) encounters a crystalline surface, it initially becomes attached to that surface as an “admolecule.” Not (yet) part of the crystal’s lattice, this admolecule diffuses across the surface until it fills an unoccupied position in the crystal lattice, or else detaches from the surface and re-enters the gas phase.

As appealing as classical nucleation theory is, there are numerous challenges. One is an effect we may call the “scale problem,” namely, that atomistic time and distance scales are vastly smaller than mesoscopic ones. Bridging that gap requires that one make judgements as to what is important, and what may be ignored. A great deal of progress has been made of late addressing this type of problem, although not specifically as applied to ice crystal dynamics.

A second challenge concerns an effect we shall term the “corner dispersal problem,” which is specific to ice crystal dynamics. It arises when a faceted ice crystal situated in a supersaturated vapor field experiences persistently higher vapor pressures at protruding facet corners. In some cases, it is believed, this results in faster growth at those corners, eventually leading to indented crystal morphologies, and even dendritic forms (e.g., snowflakes). In other cases, however, faceted morphology is maintained in the presence of such inhomogeneity – examples being the faceted ice crystals found in high-altitude cirrus clouds, and in low-pressure SEM experiments – even when facets are large enough that admolecule surface diffusion alone cannot disperse excess admolecules deposited at facet corners, in toward facet centers. There is also an analogous “rounding problem,” which applies to subsaturated conditions, in which an initially-faceted crystal in a subsaturated environment experiences persistently lower vapor pressure at protruding facet corners. In some cases, the crystal responds by adopting a rounded geometry, but in other cases faceted morphology is, again, stubbornly maintained.

The corner dispersal and rounding problems are not resolved by classical nucleation theory. To add to that criticism is the fealty (or lack of it) of the theory to what we believe we know about the molecular structure of the ice-vapor interface. It is this: when the temperature of ice rises above , both experiment and theoretical studies have shown that there are no lattice gaps directly available to an admolecule on the ice surface, because the interface is entirely covered by a quasi-liquid layer (QLL). Molecular dynamics studies have shown that this QLL thermalizes nearly every water vapor molecule that impacts the surface, on a picosecond time scale.

To bridge the gap between atomistic and mesoscale levels, therefore, it would seem that we require an atomistic model that is faithful to the existence of a QLL (above ), and that overcomes the multi-scaling problem. One such attempt was presented by some of the authors in 2016 (N2016), in the form of a dynamic quasi-liquid continuum model referred to here as QLC-1. Because that model forms the foundation for the revision presented here (“QLC-2”), we summarize that framework next.

QLC-1 frames the problem of ice surface dynamics in terms of two mesoscale variables, and (see Fig. 1), which represent the total thickness of the ice surface and the thickness of its quasi-liquid part. Time evolution of these variables is governed by a pair of reaction-diffusion differential equations that represent the three processes indicated in Fig. 1, namely, (i) exchanges (deposition and ablation) with the vapor phase, (ii) horizontal diffusion of the QLL across the ice surface, and (iii) interconversion of QLL molecules to/from the underlying ice.

|  |
| --- |
|  |
| **Figure 1**. Visual representation of mesoscale variables , , and , and processes affecting them, in QLC-1 (as well as the present revision, QLC-2) model. Dashed arrows represent processes affecting how these variables evolve over time. |

The main insight afforded by QLC-1 is that it provides a mechanism by which faceted ice crystal growth occurs. At the heart of that mechanism is a process N2016 termed “diffusive slowdown,” which can be summarized as follows:

1. At the micrometer level, the QLL can be thought of as consisting of a continuum of microstates, ranging from a thin, less-volatile microstate labeled surface I, to a thick, more-volatile one labeled surface II. The difference in these volatilities is quantified in QLC-1 as a difference in equilibrium supersaturation, .
2. In a growing ice crystal, each time a new layer forms, a new pair of these microstates appears on the surface. Since new layers typically form at facet corners (where the water vapor concentration is highest), it is also the case that the horizontal distance between newly-formed ice layers is smaller at facet corners compared to facet centers. In this paper, we designate this distance as “”; in Fig. 2(a) we see that .

|  |
| --- |
|  |
|
| **Figure 2**. An ice surface covered by QLL, as simulated by QLM-2. |

1. Horizontal diffusion moves quasi-liquid away from thicker, surface II-like regions, and toward thinner, surface I-like regions of the surface. However, because surface I accounts for a smaller fraction of the total surface area (as seen in Fig. 2(b)), diffusion has a greater proportional effect on it. As a consequence, diffusion leads to an increase in the average volatility of the surface, causing the surface as a whole to experience a net “diffusive slowdown” in its growth rate. (The reader is referred to N2016 for a quantitative version of this argument.)

A second process identified in N2016 is that diffusive slowdown is not homogeneous across a facet. Instead, in a growing ice crystal, more diffusive slowdown occurs at facet corners. The reason is as follows:

1. In regions where is small, QLL thickness gradients are large. (In Fig. 2(b), for example, it is clear that the gradient in QLL thickness at I’ is greater than at I.) It follows that, in a growing ice crystal, more diffusive slowdown occurs at facet corners.

In summary, the corners of an initially flat facet exposed to supersaturated vapor will experience an enhancement in growth rate because of increased vapor concentration above it, but that same enhancement leads to a compensating increase in diffusive slowdown because layers of ice are more tightly spaced. When these effects are in dynamic balance, the result is a steady state characterized by equal growth rates of all regions of facet. That steady state, in turn, is interpreted at the mesoscale (e.g., in a high-resolution optical or scanning electron microscopy) as faceted growth.

QLC-1 suffered from some structural deficiencies, however, of which the most important for our present purpose is that the time scale of process (iii) illustrated in Fig. 1, the interconversion of quasi-liquid and ice, was fixed relative to processes (i) and (ii). In real crystal facets, these time scales are expected to vary from facet to facet, or as a function of temperature and vapor pressure. These time scales should therefore be adjustable quantities within the theory.

Our goal in this work is to evaluate strengths and weaknesses of a revised quasiliquid continuum model for ice crystal growth and ablation designed to address this deficiency. Section 2 presents such a model, referred to here as QLC-2. Section 3 (with details given in Appendix 1) summarizes scanning electron microscopy (SEM) image processing algorithms that can be used to construct quantitative ice surface morphologies. Section 4 (with details given in Appendix 2) summarizes results from independent gas-phase simulations that can be used to inform parameterizations of QLC-2 having to do with the structure of the vapor field in contact with growing ice crystals. Section 4 presents simulations and SEM experiments in pursuit of five lines of investigation. Section 5 summarizes and discusses implications of these results in other contexts, including cirrus ice crystal morphologies and ideas from nonlinear dynamics.

1. **A revised quasi-liquid reaction-diffusion model**

The present model, QLC-2, has much in common with QLC-1, beginning with its representation of an ice surface defined by the two mesoscale variables and three processes shown in Fig. 1. The governing equations are

(1a)

(1b)

Some notes about this model are as follows, with differences between it and QLC-1 noted where relevant:

1. represents the idea that surface diffusion depends on the thickness of the quasi-liquid only; the underlying ice is considered immobile on time scales considered here.
2. is the rate at which vapor-phase water molecules collide with the quasi-liquid; it is assumed that these stick with 100% efficiency, and thermalize on a picosecond time scale. is calculated from the Hertz- Knudsen formula,

(2)

where is the equilibrium vapor pressure of water, and is water’s molar mass.

1. prescribes the thickness of quasi-liquid when it is in equilibrium with the underlying ice, according to

(3)

This formulation ensures that the QLL thickness varies continuously from the thin microstate (“surface I”) with thickness , to the thick microstate (“surface II”) with thickness .

1. Because surfaces I and II have different volatilities, the surface supersaturation at a given point on the surface (designated in Eq. 1a) is a function of both the microstate and the water vapor concentration above it. To compute , we define a variable that quantifies the degree to which a given surface is similar to surface I or II,

(4)

With this definition, surface I will have , while surface II will have . We then express the surface supersaturation as

(5)

where is a measure of the difference in the equilibrium vapor pressure of surfaces I and II, and is the supersaturation relative to surface I. We will assume here that both (a scalar quantity) and are fixed parameters of a given trajectory. Eq. 5 is at slight variance with, and simpler than, the corresponding expression in QLC-1.

1. appearing in Eq. 5 is represented by a parabolic, positive-curvature expression,

(6)

Here, is a “center reduction”, fractional the lowering of supersaturation at facet center, compared to its edges. In this work, negative values of will also be used, these serving as a “center enhancement” of supersaturation at facet centers. Although the authors of N2016 used Eq. 6, to express inhomogeneities in the surface saturation, they also used a sinusoidal form, because it was not clear at that time which shape was the more realistic. Here we focus on the parabolic form only, because that is the form motivated by vapor field simulations presented in Appendix 2, along with parameterizations of dependence of on crystal size ().

1. is a first-order relaxation constant describing the time scale at which quasi-liquid/ice equilibrium is achieved. That is, if we imagine a surface with quasi-liquid amount , then equilibration after a time occurs according to

(7)

If one takes the time derivative of Eq. 6, and assumes that is small, the second term on the right-hand side of Eq. 1b results.

Equations 7 and 1b represent the primary departure of QLC-2 from QLC-1. With this revision, we are able to specify the rate of quasi-liquid/ice equilibration relative to processes (i) and (ii). Specifying a small value for , for example, would represent the idea that quasi-liquid/ice equilibration is fast compared to those processes, while large would mean the opposite. We do not have reliable observational values of , but we do have a guidepost: because the “diffusive slowdown” mechanism for stabilization of faceted ice growth described above requires that quasi-liquid/ice equilibration be slow compared to surface diffusion, we should not be surprised if we find that large leads to stable growth dynamics. We return to this topic below.

1. **ESEM/GNBF retrievals**

Environmental SEM of imaging of ice crystals has seen considerable development in recent years, including the ability to image actively growing and ablating crystals by manipulating the temperature and pressure inside an SEM chamber. In tandem with those developments are computer codes for generating quantitative surface morphologies using a Gauss-Newton in a Bayesian Framework (GNBF) algorithm. The combination – a process we will refer to here as “ESEM/GNBF retrieval” – provides opportunities for vetting model predictions of surface morphology against experiment at resolutions that are not quite commensurate with one another, but approaching that level. Details are given in Appendix 1.

1. **Gas-phase simulations**

QLC-2 requires numerous parameterizations. Some of these were provided, in N2016, by molecular dynamics simulations, and some by experimental observations. Unexploited in N2016, however, is the power of vapor-phase simulations to constrain properties of the vapor field overlying a given ice surface. Details are given in Appendix 2.

1. **Results**

Here we describe the results of five lines of investigation we have pursued, each focusing on a particular topic or question. ESEM/GNBF and gas-phase modeling results are presented alongside, as relevant.

*I. Effect of variation in the time scale of modeled ice-quasiliquid equilibration*

[still working on this, preliminary work says that bigger stabilizes the formation of steady states, but otherwise has little effect on the shape of the steady-state profiles] … See Fig. 3 …

|  |
| --- |
|  |
| **Figure 3**. Effect of varying the ice-QLL equilibration timescale |

*II. Curvature of faceted surfaces undergoing growth and ablation*

Figure 4 shows a modeled ice crystal surface under growing and ablating conditions. The growth scenario on the left of the figure resulted from supersaturated water vapor concentrations, distributed as shown in Fig. 4(a). Figures 4(b) and 4(c) show that these conditions lead to steady state, “V”-shaped profiles, in which the surface is dominated by primarily surface I – like microstates. This scenario exhibits more tightly bunched (smaller ) at facet boundaries, which in turn (as described above in the summary of diffusive slowdown) leads to a net increase in volatility of the surface as a whole, hence faceted growth.

The ablating scenario on the right of Fig. 4 resulted from subsaturated water vapor amounts, distributed as shown in Fig. 4(d). Figures 4(e) and 4(f) show that these conditions also lead to steady state, although in this case the profile is “” shaped (i.e., rounded), the surface is dominated by surface I – like microstates, and the layer bunching leads to *reduced* volatility of the surface near the corners, hence faceted ablation.

The model results shown in Fig. 4 also suggest the following general pattern: growing ice facets possesses convex curvature, whereas ablating ice facets possess concave curvature. A useful metric for describing the curvature of steady state profiles such as those appearing in Fig. 4 is the local slope of the surface. Here we quantify that slope as a mean horizontal distance between successive molecular layers, defined as

(8)

For example, the growing facet profile on the left of Fig. 4 is characterized by , whereas the ablating facet profile on the right is characterized by .

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
| **Figure 4**. Stabilization of growing (left panels) and ablating facets (right panels). | |

It is well-known that real ice crystals exhibit faceted growth, but are they also capable of faceted ablation? Figure 5 displays ESEM images of an ice crystal observed under growing and ablating conditions. Since the ablating crystal retains its flat surface, we can conclude that faceted ablation has indeed occurred. The figure shows, moreover, that faceted ablation occurs even when the surface is rough (e.g., the prismatic facets in the figure). In fact, we observe faceted ablation quite frequently in ESEM images of ablating ice crystals.

|  |  |
| --- | --- |
|  |  |
| **Figure 5**. An ice crystal under growing (left) and ablating (right) conditions. | |

Do real faceted ice crystals exhibit curvature like that indicated by the QLC-2 predictions of Fig. 4? Fig. 6(a) displays an ESEM image of a crystal and a GNBF construction of a portion of its basal facet. The crystal is known to be growing, since subsequent images taken of this crystal revealed expanding boundaries against the metal substrate to which the crystal is attached. The GNBF construction, displayed in Fig. 6(b), reveals a distinct convexity, on the order of 1000s of layers over the horizontal span analyzed () … which corresponds to .

Turning to ablation, Figs. 6(c-d) show an ESEM image of an ablating, faceted crystal, and its GNBF-constructed surface. *The GNBF reconstruction hopefully reveals concavity ….* .*]*

|  |  |
| --- | --- |
|  | Need image |
|  | Need image |
| **Figure 6**. SEM image of a growing crystal (left) and an ablating crystal (right) …. | |

Thus, the pattern predicted by QLC-2 is borne out: growing ice facets do indeed exhibit facet convexity, while ablating ice facets exhibit facet concavity – although we hasten to point out that the values are far smaller in the observations than in the model.

*III. Characteristic length scales of growing and ablating surfaces*

Here we investigate the possibility that growing and ablating ice crystals possess intrinsic length scales. Focusing first on observations, we note that a distinct growth/ablation asymmetry appears in the roughening evident in the ESEM images displayed in Fig. 5. …

Figure 7 shows the dependence of the mean horizontal layer separation, (introduced above), on the dimensionless parameter , defined by

(9)

where (as described above) is the surface diffusion coefficient, is the edge length of the crystal, and is the kinetic deposition velocity. In constructing Fig. 7, components of ranged in value as follows:

* to ;
* to ; and
* to .

|  |  |
| --- | --- |
| **Figure 7**. Mean horizontal layer separation () as a function of parameter for a fixed corner supersaturation of . |  |

It is clear from Fig. 7 that a good parameterization is

(10)

where best-fit parameters are , . Similar analysis of ablating wavelengths yields

(11)

with and .

Figure 8 shows values of for a range of corner supersaturations (), holding parameter constant. We see that when conditions begin to become supersaturated, starts off at about and declines monotonically, ultimately leading (we surmise) to hollowed or dendritic growth at high supersaturations. On the left-hand side are shown results when conditions are subsaturated. We see that under these conditions, starts much higher than on the supersaturated side, at about , and declines monotonically with increasing subsaturation, ultimately leading (we surmise) to facet rounding.

|  |
| --- |
|  |
| **Figure 8**. Mean horizontal layer separation () as a function of corner supersaturation . |

We have carried out a best-fit analysis of the supersaturated-condition points on the right-hand side of Fig. 8, using an exponential, which when combined with the previous expressions (Eqs. 9 and 10) yield

Combining the above expressions, the functional forms are

(12)

and

(13)

where , with , , and .

Figure 8 shows the results of applying this functional form to the numerical data, using best-fit parameters , , , The fit is pretty good … And what it means is, we can predict steady-state properties over a very big range of physical variables (crystal length, surface diffusivity, , and imposed supersaturations).

Thus, both theory and experiment support the existence of characteristic length scales as a function of super/subsaturation. We hasten to add that the scale of these phenomena is vastly different ….

*IV. Resilience of steady states*

In ESEM experiments, facets exhibit a certain resilience, in that an initially faceted surface, after it is roughened by some perturbation (e.g., by higher temperature or a lower supersaturation), can usually be restored to its initial smooth faceted state after the initial conditions are restored. An example is shown in Fig. 9

|  |
| --- |
|  |
| **Figure 9**. SEM image of a rough facet restored to smoothness. |

What resilience, if any, is exhibited by modeled facets? Figure 10 is an examination of this question.

|  |
| --- |
|  |
| **Figure 10**. Examination of model facet resilience. Curves in the upper-left graph show timelines of that characterize the surface when subjected to the supersaturation curves shown in the upper-right insets, over the indicated time intervals. |

The sequence of images in Fig. 10 begins with an initially-flat profile, which is then subjected to the following sequence:

1. From to , the surface is been exposed to the supersaturated water vapor curve shown in the inset located in the upper right part of the figure. By , the profile has evolved to the faceted, steady-state profile labeled “A”.
2. From to , a perturbation is introduced in the form of the subsaturated water vapor curve shown in the inset located in the middle-right part of the figure. During this time, the surface evolves into the highly perturbed state labeled “B”.
3. From to , the initial supersaturated water vapor regime is restored. During this time, the surface recovers its pre-perturbation faceted profile labeled “C”.

Conclusion … qualitatively, QLC-2 exhibits resilience analogous to that of real crystal facets, such as the one shown in Fig. 9. We should hasten to add, however, the perturbed states exhibit vastly different vertical scales. That is, in Fig. 10, the perturbed profile seen at , caused by the perturbation begun at , is non-faceted (i.e., “rough”) only on the order of a few dozen monolayers of ice, whereas the actual crystal shown in Fig. 9 exhibits ridge-to-valley depths on the order of thousands of monolayers.

1. **Summary and discussion**

The QLC-2 model presented here offers significant technical improvements over the QLC-1 model reported in N2016, in that its numerical solutions are more stable, and its parameterizations are better constrained by independent numerical simulations of the overlying vapor field, i.e., the dependence of the center reduction parameter () on crystal size ().

More fundamentally, QLC-2’s equations of motion embody a more faithful atomistic representation of ice/QLL freeze/melt equilibration. That connection, in turn, enables the theory to represent a more unified picture of ice surface dynamics than has previously been possible. The model predicts, for example, facet convexity and concavity under growth and ablation conditions (respectively), a pattern that is echoed in reconstructions of SEM-grown ice crystals. The model also provides a mechanism by which real ice crystals resist dendritic geometries when subjected to supersaturation conditions (i.e., faceted growth), and how they also resist rounding when subjected to subsaturation conditions (faceted ablation). It is, essentially, a theory of faceting that rests fundamentally on atomistic variations in the thickness and volatility of the quasi-liquid layer that are exposed as a crystal grows or ablates.

The model also predicts a certain resilience to perturbations – also echoed by real ice crystals – in that faceted surfaces that have been disrupted in some way recover when conditions favorable to faceting are restored.

QLC-2 also makes some interesting quantitative predictions about intrinsic characteristic distances – defined here as the mean distance between adjacent layers of ice, . We find that this distance separates mathematically as a product of two functions: one is a linear function of dimensionless variable, , and the other is a highly nonlinear function of the surface supersaturation. We find that the latter is an asymmetrical function of the surface supersaturation, such that growth conditions lead to smaller values of , subsaturation conditions lead to larger values of , for comparable departures from saturation. We have no ready explanation for this behavior, but we do note that SEM observations are consistent with these observations in the sense that ridges in rough facets of growing crystals are spaced more closely to one another than in rough ablating crystals. As noted previously, this coincidence is subject to a very big caveat, namely, that the depths of these characteristic distances are vastly different in QLC-2 compared to experiment: in the former it is a few monolayers, while in the latter it is thousands of monolayers.

It is of interest to note implications of these results regarding morphological changes cirrus cloud particles undergo as they fall through Earth’s atmosphere. Such crystals will encounter increased pressure, hence smaller , but also increased temperatures (unless there is an atmospheric inversion), hence larger . When the effect of increased pressure dominates, QLC-2 predicts enhanced growth at facet corners, hence greater facet convexity, and therefore a greater propensity toward hollowed crystal structures. When the effect of increased temperature dominates, however, we can expect more regular hexagonal shapes. Although there is no single observational datum that would help us resolve these predictions, we can comment that in exceptionally cold regions (such as the Antarctic Plateau), or even in mid-latitudes where high-altitude cryo-capture of ice crystals on ground-launched balloons is possible, observations have shown that cirrus clouds are frequently hollowed, suggesting dominance of increased pressure.

We can also expect that because different facet types (basal, prismatic, or pyramidal) have distinct underlying crystal cell structures, their quasi-liquid properties will also be distinctive. Exploratory numerical studies varying the thickness of a single “layer” of ice has shown that a proportional increase in results. Preliminary numerical experiments varying and have shown that …

A separate speculation concerns the observation that the dependence of values exhibited by QLC-2 is the same as in Turing patterns. In one sense this should come as no surprise, since Turing’s theory, like QLC-2, is based on a reaction-diffusion equation. But there are also very big differences, including the fact that Turing’s analysis proceeds from an analysis of sensitivity to perturbations to an initially homogeneous distribution of chemical species, whereas the patterns in QLC-2 emerge as steady states of the equations of motion.

Finally, we note that the atoms-to-mesoscale approach represented in QLC-2 is not as fully integrated as we would like it to be. In particular, if the temperature and pressure dependence in the vapor field calculations could be integrated with QLC-2, we would be able to take into account feedbacks that are not present within QLC-2 alone. The problem is the difference in time scales: vapor equilibration takes place hundreds of times faster than QLC-2 surface calculations. [But maybe Jake’s heterogeneous multiscale approach is the way forward on that].

**Appendix 1 – ESEM/GNBF retrieval**

Here is a summary of key equations and constraints in the ESEM/GNBF retrieval …

**Appendix 2 – Vapor phase simulations**

Simulation of the water vapor partial pressure, , in the space surrounding a square-shaped crystal shown in Fig. A1(a), was achieved by integrating the two-dimensional diffusion equation

(A1)

where is the diffusion coefficient of water vapor through air, which is computed by (based on a fit to data given in the Engineering Toolbox),

(A2)

|  |  |
| --- | --- |
| (a) | (b) |
|  |  |
| **Figure A1**. (a) Simulation of vapor partial pressures, , in units , around a growing ice crystal. The black-outlined box at the center indicates the surface of the crystal, in this case spanning . Contours outside the crystal show values of at steady state. The entire simulation space spans in both dimensions. At , far-field partial pressures are fixed at . (b) along the surface of the crystal. | |

The last term on the right of Eq. A1 implements Neumann boundary conditions that represent depletion of water vapor in the layer adjacent to the crystal surface due to crystal growth. Variables appearing in this term are:

* + is the specified growth rate of the ice surface; it equals zero except at the surface of the ice crystal, where its value is prescribed as (typically ).
  + is the mass density of ice, ;
  + is the molar mass of water, ; and
  + is the spatial discretization in the -direction (typically, ).

Dirichlet conditions, representing the far-field vapor concentration (), are imposed at the outside boundary of the simulation space.

Integration over time was performed using Euler’s method, i.e., with time steps , and distance intervals (which equals ). That is, at each time step, changes in were computed by

(A3)

Parameters for the integration are given in Table A1. The resulting vapor concentration contours shown in Fig. A1(a) are seen to decrease with proximity to the crystal, as expected since the growing crystal is drawing water vapor out of the surrounding air. This reduction is greater at facet center compared to facet corners, in a roughly parabolic fashion, as shown in the profile in Fig. A1(b), thus justifying the form given in Eq. 6 of the text, where the “center reduction” is defined by

(A4)

For example, in A1(b), .

|  |  |
| --- | --- |
| **Table A1. Parameters for simulation of the vapor field around a growing, square-shaped ice crystal** | |
| Simulation space dimensions |  |
| Time step for integration |  |
| Time interval for integration |  |
| Spatial discretization |  |
| Diffusion coefficient at , |  |
| Ambient temperature |  |
| Ambient pressure |  |
| Diffusion Temperature-correction exponent |  |
| Diffusion coefficient under ambient conditions |  |
| Far-field water vapor partial pressure |  |
| Far-field water vapor supersaturation |  |
| Mass density of ice |  |
| Growth rate of ice surface |  |
| Slope parameter for under ambient conditions: |  |

Other simulation results (not shown) exhibit the expected property that higher far-field vapor concentrations, , lead to higher . The consequence of this should be that crystals would grow faster under such conditions, and therefore offset some of the increased , but that (negative) feedback is not built into Eq. A1, because is a fixed parameter.

Figure A2 shows simulation results as a function of the crystal edge length. Fig. A2(a) shows that the steady-state concentration of water vapor at crystal corners declines with increasing crystal size. This is expected, since a larger growing crystal means there is more crystal surface area drawing water vapor out of the air.

|  |  |
| --- | --- |
| (a) | (b) |
|  |  |
| **Figure A2**. (a) Steady-state supersaturation, , at crystal corners, as a function of crystal size. (b) Percent reduction in at facet center as compared to at crystal corner (Eq. A4). | |

Fig. A2(b) shows the percent reduction in surface vapor concentration at facet center relative to facet corner (see the definition of in Eq. A4). is seen to be a smoothly increasing function of crystal size, well-described by

(A5)

Other simulations (not shown) show that higher far-field vapor concentrations () do not greatly alter this parameterization. Also, this parameterization is entirely insensitive to . Because of this invariance, is more useful than for describing the concentration of water vapor over a growing ice crystal facet.

Yet other simulations show that this parameterization is greatly affected by temperature (which reduces considerably), and by the ambient pressure (which increases considerably). Lower ambient temperatures and higher ambient pressures are both associated with smaller vapor diffusion coefficients, . Smaller *,* in turn, manifests as a more steeply curved profile across the crystal surface, compared to the example shown in Fig. A1(b): it is harder for water vapor to diffuse across the facet surface when is small. The expected consequence is that lower ambient temperatures and higher ambient pressures will increase the tendency for excess growth at facet corners, ultimately leading to highly convex, indented, hollowed crystal morphologies, and even, at sufficiently small , dendritic forms such as snowflakes.

**Appendix 3 – Numerical considerations**

Python, accelerated with Numby. Code and data are available on Github.

References:

The temperature dependence of the diffusion coefficient was based on data from <https://www.engineeringtoolbox.com/air-diffusion-coefficient-gas-mixture-temperature-d_2010.html>